COMPUTATION OF LYAPUNOV CHARACTERISTIC EXPONENTS SPECTRUM BY THE COMPOUND MATRIX METHOD

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Abstract

The Compound Matrix Method, that have been used for the numerical integration of differential equations in the field of hydrodynamics [1,2], is applied for the numerical computation of the spectrum of Lyapunov Characteristic Exponents (LCEs) for dynamical systems of various degrees of freedom. The method overcomes the necessity of applying an orthonormalization procedure, that preserves the linear independence of the deviation vectors, during the numerical integration. This is achieved by translating the considered vectors into areas, volumes and hyper-volumes. The Fermi-Pasta-Ulam (FPU) lattices were considered and studied for various degrees of freedom. The obtained results are compared with the ones produced by the well-known standard method of [3].

FPU β-lattice

The model describes a chain of N particles with nearest-neighbor interaction. It's Hamiltonian is :

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=0}^{N} \left[\frac{(q_{i+1}-q_i)^2}{2} + \frac{\beta \cdot (q_{i+1}-q_i)^4}{4} \right] \quad \text{with :} \quad q_0 = q_{N+1} = p_0 = p_{N+1} = 0$$

The vector of an orbit in the 2N dimensional phase space is :

$$\vec{x}(t) = (\vec{q}(t), \vec{p}(t)) = (q_1(t),...,q_N(t), p_1(t),...,p_N(t))$$

The time evolution of the orbit is governed by the Hamilton equations of motion :

$$\dot{\vec{x}} = \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_N}{\partial p} \\ -\frac{\partial H_N}{\partial q} \end{bmatrix} \quad \text{with :} \quad \begin{cases} \dot{q}_i = \frac{\partial H_N}{\partial p_i} = p \\ \dot{p}_i = -\frac{\partial H_N}{\partial q_i} = q_{i+1} - 2q_i + q_{i-1} + \beta\left[(q_{i+1}-q_i)^3 - (q_i - q_{i-1})^3\right] \end{cases}$$

A deviation vector evolving on the tangent space is of the form :

$$\vec{w}(t) = (\delta\vec{q}(t), \delta\vec{p}(t)) = (\delta q_1(t),...,\delta q_N(t), \delta p_1(t),...,\delta p_N(t))$$

It's time evolution is governed by the variational equations :

where :

$$\dot{\vec{w}}(t) = \begin{bmatrix} \dot{\delta\vec{q}} \\ \dot{\delta\vec{p}} \end{bmatrix} = A(t) \cdot \vec{w}(t) = \begin{bmatrix} 0_N & I_N \\ -D^2_{H_N}(\vec{q}(t)) & 0_N \end{bmatrix} \cdot \begin{bmatrix} \delta\vec{q} \\ \delta\vec{p} \end{bmatrix}$$

$$-D^2_{H_N}(\vec{q}(t)) = -\frac{\partial^2 H}{\partial q_i \cdot \partial q_j} = \begin{cases} -2-3\beta\left[(q_{i+1}-q_i)^2 + (q_i - q_{i-1})^2\right], & i = j \\ 1+3\beta(q_i - q_j)^2, & abs(i-j) = 1 \\ 0, & abs(i-j) > 1 \end{cases}$$

Compound Matrix Method

The method applies the theory of exterior algebra and wedge product in order to create ordinary differential equations for the time evolution of areas, volumes and hyper-volumes constructed by the considered deviation vectors. This is done by appropriate combinations of the coordinates of the deviation vectors according to the lexicographical order. For the case of N=2 particles, there are four possible different combinations of the k=4 deviation vectors. For k=1 we get the usual vector, whose evolution is defined by the variational equations. The wedge product of k=2 deviation vectors defines a two-dimensional surface, while the wedge product of k=3 vectors describes a three-dimensional volume. Finally, in the k=4 case, the wedge product corresponds to the determinant of the 4x4 matrix having as elements the coordinates of the 4 deviation vectors. The equations defining the evolution of these quantities are numerically integrated by an extension of the symplectic integration techniques used for the integration of the Hamilton equations of motion according to the so-called tangent map technique [4]. Following this approach, we do not need to apply the orthonormalization procedure used in the standard method of [3]. The following flowchart shows the successive steps of the method :

$$\dot{\vec{w}}_i(t) = A(t) \cdot \vec{w}_i(t) \quad \Rightarrow \quad \dot{\vec{y}}^{(k)}(t) = A^{[k]}(t) \cdot \vec{y}^{(k)}(t)$$

where :

$$i = 1,2..... \leq 2N$$

$$k = 1,2..... \leq d = \binom{2N}{k} = \frac{2N!}{k!(2N-k)!}$$

$y^{(k)}, A^{[k]}$: multiplicative and additive compound matrices respectively

Considering the case of N=2 particles, the various cases of k additive matrices, according to [5,6,7] are :

For $A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ H_{q_1q_1} & H_{q_1q_2} & 0 & 0 \\ H_{q_2q_1} & H_{q_2q_2} & 0 & 0 \end{bmatrix}$ then $k = 1 \Rightarrow A^{[1]}(t) = A(t)$

$$k = 2 \quad \Rightarrow \quad A^{[2]}(t) = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ H_{q_1q_2} & 0 & 0 & 0 & 0 & 0 \\ H_{q_2q_2} & 0 & 0 & 0 & 0 & 1 \\ -H_{q_1q_1} & 0 & 0 & 0 & 0 & -1 \\ -H_{q_2q_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -H_{q_2q_1} & H_{q_1q_1} & -H_{q_2q_2} & H_{q_1q_2} & 0 \end{bmatrix}$$

$$k = 3 \Rightarrow A^{[3]}(t) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -H_{q_2q_2} & H_{q_1q_2} & 0 & 0 \\ H_{q_2q_1} & -H_{q_1q_1} & 0 & 0 \end{bmatrix} \qquad k = 4 \Rightarrow A^{[4]}(t) = \left|A(t)\right| = 0$$

Application of the Compound Matrix Method

We compare the LCEs obtained by the standard method of [3] with the ones found by the compound matrix method for various numbers of particles. The fourth order symplectic integrator SABA2C [4] is used for the numerical integrations. The time step was fixed at dt=0.01 and the maximum integration time was set to be tmax=100.000. The energy of the system is fixed at E=10.0.

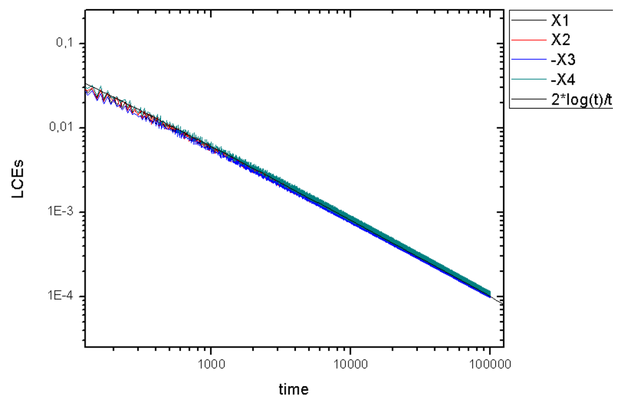

Figure 1.a : Computation of the spectrum of LCEs (X1>X2>X3>X4) of a regular orbit for N=2 and Energy=10.0, by the method of [3].

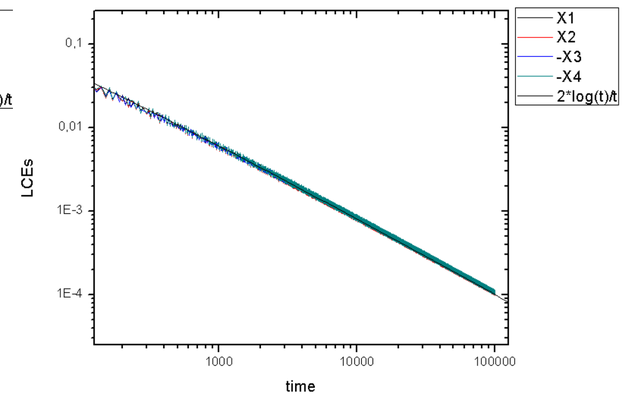

Figure 1.b : Computation of the spectrum of LCEs of the regular orbit of Fig. 1.a, by the compound matrix method.

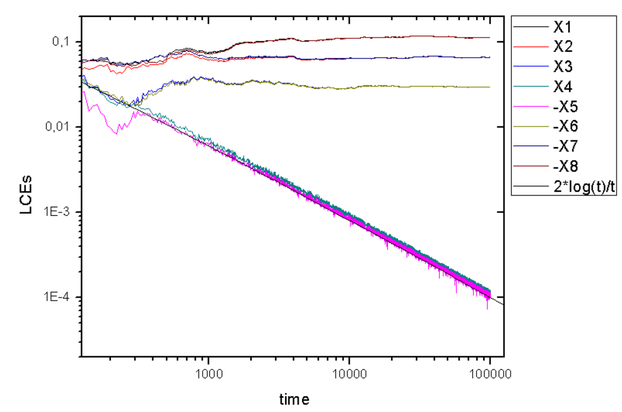

Figure 2.a : Computation of the spectrum of LCEs of a chaotic orbit for N=4 and Energy=10.0 by the method of [3].

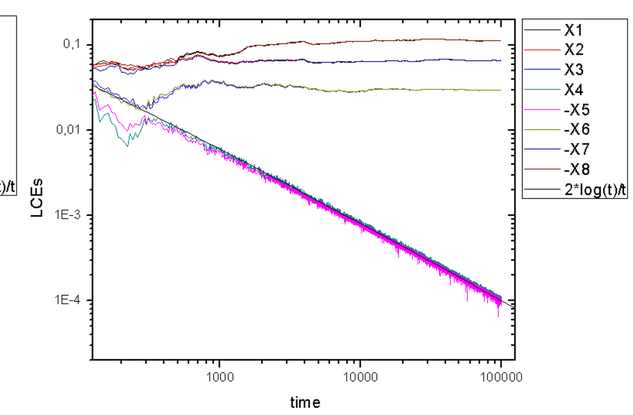

Figure 2.b : Computation of the spectrum of LCEs of the chaotic orbit of Fig 2.a, by the compound matrix method.

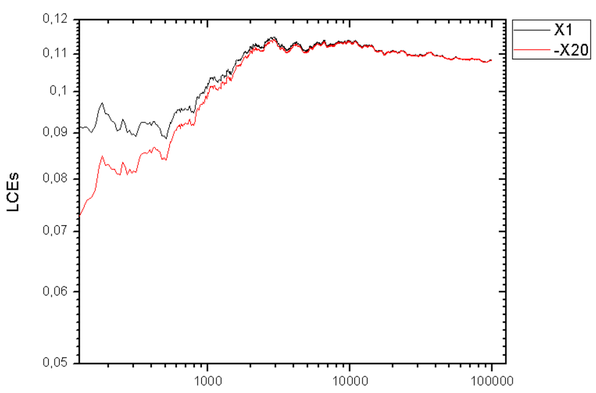

Figure 3.a : Computation of the mLCE (X1) and its inverse (X20) of a chaotic orbit for N=10 and Energy=10.0, by the method of [3].

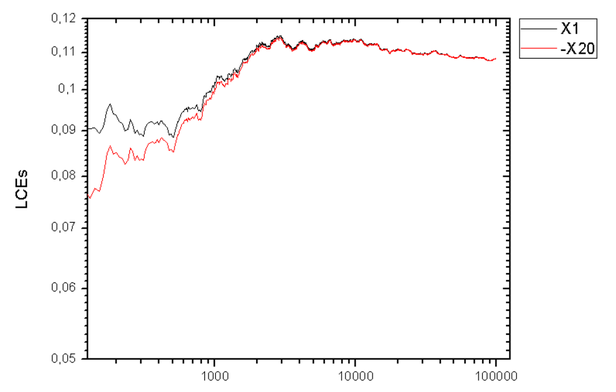

Figure 3.b : Computation of the mLCE (X1) and its inverse (X20) of the chaotic orbit of Fig 3.a, by the compound matrix method.

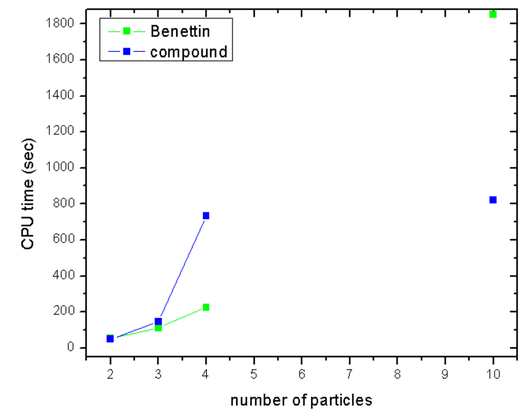

Figure 4 : CPU times needed for the numerical computation of the spectrum of LCEs for N=2,3,4 particles and for the computation of the mLCE (X1) and its inverse (X20) for N=10 particles, applying the method of [3] and the compound matrix method.

Conclusions

The compound matrix method provides an alternative, efficient approach for the computation of the whole spectrum of LCEs. It can also be used for the computation of individual LCEs. Its basic feature is that it produces differential equations that describe the evolution of hyper-volumes instead of individual deviation vectors as the usual variational equations do.

References

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